if $|c_0| z$ is sufficiently large. This demonstrates that the numerical factor on the right side of (56) cannot be increased with zeros restricted by (57).

ACKNOWLEDGMENT

The authors wish to thank W. D. Warters for suggesting the present problem.

REFERENCES

[1] H. E. Rowe and D. T. Young, "Transmission distortion in multimode random waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 349–365, June 1972.
[2] H. E. Rowe, "Approximate solutions for the coupled line equations," *Bell Syst. Tech. J.*, vol. 41, pp. 1011–1029, May 1962.
[3] H. E. Rowe and W. D. Warters, "Transmission in multimode waveguide with random imperfections," *Bell Syst. Tech. J.*, vol. 41, pp. 1031–1170, May 1962.
[4] A. Papoulis, *The Fourier Integral and Its Applications*. New York: McGraw-Hill, 1962, pp. 204–212.

Statistical Coupled Equations in Lossless Optical Fibers

BRUNO CROSIGNANI, BENEDETTO DAINO, AND PAOLO DI PORTO

Abstract—**The problem of deriving sets of statistical coupled equations for the second and fourth moments of the mode amplitudes in a fiber with mode coupling is considered, starting from the deterministic coupled wave equations describing an electromagnetic field propagating in a lossless fiber. Our results extend the work of Marcuse, and, in particular, allow one to deduce sets of equations for quantities which describe the cross correlation between different modes. Furthermore, we obtain new results regarding the variances and cross correlations of the power in the modes (fourth-order amplitude statistics).**

I. INTRODUCTION

An electromagnetic wave propagating in an optical fiber can be described by means of a set of coupled differential equations for the amplitudes of the modes supported by the guide. The coupling terms are, in particular, associated with the deviations of the fiber from the ideal structure pertaining to a regular geometrical form and refractive index distribution. In many situations, these imperfections are distributed in a complicated fashion along the guide, so that it is difficult to determine the spatial behavior of the coefficients of the fundamental equations for a given fiber, and, also, if they are known, it is practically impossible to deduce an analytical solution.

In order to circumvent these difficulties, it is useful to introduce a statistical ensemble of fibers possessing small random deviations from a common ideal structure [1], [2]. The problem is then to obtain simple equations for the ensemble averages of quantities describing either the evolution of each propagation mode, or the interaction between different modes. The perturbative approach [1], [2] allows one to derive a closed system of equations for the ensemble averages of the powers of the coupled modes, also taking into account losses due to small coupling with radiation modes.[1]

The behavior of the variance of the power has also been investigated [2], in the limit of a large number of coupled modes, thus enabling one to give an estimate of the applicability of the results of the statistical theory to a single fiber.

In this paper, we wish to introduce an analytical approach, which slightly improves the procedure followed in [1] and [2], and allows us to obtain, for a lossless optical fiber, in a straightforward way, beyond the equations for the powers, closed systems of coupled equations for ensemble averages of products of amplitudes of different modes. Furthermore, we obtain a closed system of equations connecting the averages of the power squares to those of the products of different mode powers.

As a particular application, we estimate the normalized variance of the asymptotic power distribution, which turns out to depend on the number of coupled modes.

II. COUPLED POWER EQUATIONS

We start from the relevant deterministic wave equations valid for the single fiber, which couple forward-traveling guided modes and are obtained from the general theory [4] by neglecting coupling with backward-traveling modes and radiation modes. For a steady-state situation, they read

Manuscript received June 18, 1974; revised November 27, 1974.
B. Crosignani and P. Di Porto are with the Laser Laboratory, Fondazione Ugo Bordoni, I.S.P.T., Rome, Italy, and the Department of Electrical Engineering, University of Rome, Rome, Italy.
B. Daino is with the Laser and Optoelectronic Division, Fondazione Ugo Bordoni, I.S.P.T., Rome, Italy.

[1] For good sources on the treatment of stochastic equations, see [3].

$$dc_\mu/dz = \sum_{\nu=1}^{n} A_{\mu\nu} c_\nu, \qquad (\mu = 1,2,\cdots,n) \tag{1}$$

where n is the number of coupled modes, c_ν represents the amplitude of the forward-traveling νth mode, and z labels the distance from the fiber input. The coupling coefficient $A_{\mu\nu}$ can be expressed as

$$A_{\mu\nu}(z) = K_{\mu\nu}(z) \exp[i(\beta_\mu - \beta_\nu)z] \tag{2}$$

where β_μ represents the propagation constant of the μth mode at the fixed frequency ω of the field, and $K_{\mu\nu}$ is, at least for weakly guiding fibers, a purely imaginary quantity depending on the irregularities of the geometrical structure and of the refractive index of the guide itself [4]. By using (1) and its complex conjugate written for the kth mode, one easily obtains

$$(d/dz)(c_k^* c_\mu) = \sum_{\nu=1}^{n} (A_{\mu\nu} c_\nu c_k^* + A_{k\nu}^* c_\nu^* c_\mu) \tag{3}$$

which can be formally integrated to yield, after two successive iterations,

$$\begin{aligned} c_k^*(z)c_\mu(z) = {} & c_k^*(0)c_\mu(0) + \sum_{\nu=1}^{n} \int_0^z dz' [A_{\mu\nu}(z')c_k^*(0)c_\nu(0) \\ & + A_{k\nu}^*(z')c_\nu^*(0)c_\mu(0)] + \sum_{\nu=1}^{n}\sum_{\lambda=1}^{n} \int_0^z dz' \\ & \cdot \left\{ A_{\mu\nu}(z') \int_0^{z'} dz'' [A_{\nu\lambda}(z'')c_k^*(z'')c_\lambda(z'') \right. \\ & \left. + A_{k\lambda}^*(z'')c_\lambda^*(z'')c_\nu(z'')] \right\} \\ & + \sum_{\nu=1}^{n}\sum_{\lambda=1}^{n} \int_0^z dz' \left\{ A_{k\nu}^*(z') \int_0^{z'} dz'' \right. \\ & \cdot [A_{\mu\lambda}(z'')c_\nu^*(z'')c_\lambda(z'') \\ & \left. + A_{\nu\lambda}^*(z'')c_\lambda^*(z'')c_\mu(z'')] \right\}. \end{aligned} \tag{4}$$

In order to make the problem analytically tractable, it is useful to introduce a statistical ensemble constituted by similar fibers possessing different irregularities, which are described by means of the correlation functions

$$\begin{aligned} \langle K_{\mu\nu}(z')K_{\lambda\rho}(z) \rangle &= \langle K_{\mu\nu}(z)K_{\lambda\rho}(z') \rangle \\ &= \langle K_{\mu\nu}(z - z')K_{\lambda\rho}(0) \rangle \end{aligned} \tag{5}$$

where $\mu,\nu,\lambda,\rho = 1,2,\cdots,n$, the symbol $\langle\cdots\rangle$ means ensemble average, and the ensemble of fibers is assumed to be spatially homogeneous in the z direction. Accordingly, the significant quantities are averages, over the same ensemble, of the corresponding deterministic quantities. If we now observe that

$$\langle A_{\mu\nu}(z')c_k^*(0)c_\nu(0) \rangle = \langle A_{\mu\nu}(z') \rangle c_k^*(0)c_\nu(0) = 0, \qquad (\mu,\nu,k = 1,2,\cdots,n) \tag{6}$$

due to the absence of statistical uncertainty in the value of the field at the beginning of the fiber and to the relation

$$\langle A_{\mu\nu}(z') \rangle = 0 \tag{7}$$

we can approximately write, after averaging both sides of (4),

$$\begin{aligned} (d/dz)X_{k\mu}(z) = \sum_{\nu=1}^{n}\sum_{\lambda=1}^{n} & \left\{ X_{k\lambda}(z) \exp[i(\beta_\mu - \beta_\lambda)z] \right. \\ & \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\lambda - \beta_\nu)\mathfrak{z}] \\ & \cdot \langle K_{\mu\nu}(\mathfrak{z})K_{\nu\lambda}(0) \rangle \\ & + X_{\lambda\nu}(z) \exp[i(\beta_\mu - \beta_k + \beta_\lambda - \beta_\nu)z] \\ & \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_k - \beta_\lambda)\mathfrak{z}] \\ & \cdot \langle K_{\mu\nu}(\mathfrak{z})K_{k\lambda}^*(0) \rangle \\ & + X_{\nu\lambda}(z) \exp[i(\beta_\mu - \beta_k + \beta_\nu - \beta_\lambda)z] \\ & \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\lambda - \beta_\mu)\mathfrak{z}] \\ & \cdot \langle K_{k\nu}^*(\mathfrak{z})K_{\mu\lambda}(0) \rangle \\ & + X_{\lambda\mu}(z) \exp[i(\beta_\lambda - \beta_k)z] \int_0^\infty d\mathfrak{z} \\ & \left. \cdot \exp[i(\beta_\nu - \beta_\lambda)\mathfrak{z}] \langle K_{k\nu}^*(\mathfrak{z})K_{\nu\lambda}^*(0) \rangle \right\}, \\ & (k,\mu = 1,2,\cdots,n) \end{aligned} \tag{8}$$

where

$$X_{k\mu}(z) = \langle c_k^*(z)c_\mu(z) \rangle, \qquad (k,\mu = 1,2,\cdots,n) \tag{9}$$

and use has been made of the factorization hypothesis

$$\begin{aligned} \langle K_{\mu\nu}(z')K_{k\lambda}(z'')c_\alpha^*(z'')c_\beta(z'') \rangle \\ = \langle K_{\mu\nu}(z')K_{k\lambda}(z'') \rangle X_{\alpha\beta}(z'') \end{aligned} \tag{10}$$

for all values of the indices and for $z' \geq z''$.

In order to determine the limits of validity of (8), we first note that (10) is obviously satisfied for $z' - z'' > z_0$, where z_0 defines the "correlation length" of the set of statistical variables $K_{\mu\nu}$, that is, the smallest length for which

$$\langle K_{\mu\nu}(\mathfrak{z})K_{k\lambda}(0) \rangle = 0, \qquad \text{for } \mathfrak{z} > z_0 \tag{11}$$

for all values of the indices. In the significant range $z' - z'' < z_0$, the approximation involved in (10) depends on the variation of the X's over a length of order z_0, so that

$$\frac{\langle K_{\mu\nu}(z')K_{k\lambda}(z'')c_\alpha^*(z'')c_\beta(z')\rangle - \langle K_{\mu\nu}(z')K_{k\lambda}(z'')\rangle X_{\alpha\beta}(z')}{\langle K_{\mu\nu}(z')K_{k\lambda}(z'')\rangle X_{\alpha\beta}(z')} = 0(nKz_0) \quad (12)$$

where 0 is the order symbol and $K = \max \langle | K_{\nu\lambda} | \rangle$ over all values of the indices, as can be seen by observing from (1) that the absolute value of the relative variation of the c_ν over z_0 does not exceed nKz_0 in most cases. Thus the relative approximation of (8) is of the order nKz_0.

Let us now make the further assumption, which can be verified *a posteriori*, of no significant variation of the X's over distances of the kind $1/(\beta_\mu - \beta_\lambda)$, $1/(\beta_\mu - \beta_\nu + \beta_\lambda - \beta_k)$, unless

$$\beta_\mu - \beta_\lambda = 0 \quad (13)$$

$$\beta_\mu - \beta_\nu + \beta_\lambda - \beta_k = 0 \quad (14)$$

which only holds if $\mu = \lambda$, and if either $\mu = \nu$, $\lambda = k$, or $\mu = k$, $\lambda = \nu$.[2] That is to say, the X's are slowly varying functions of z, compared with the sine and cosine functions appearing in (8). Thus, since integrals over z of rapidly oscillating terms can be neglected with respect to integrals of slowly varying functions, we are allowed to consider only the terms of (8) fulfilling (13) or (14). Within this approximation, we obtain from (8) the two closed systems of equations

$$\begin{aligned}(d/dz)X_{kk}(z) = {} & X_{kk}(z) \sum_{\nu=1}^{n} \int_{-\infty}^{+\infty} d\mathfrak{z} \exp[i(\beta_k - \beta_\nu)\mathfrak{z}] \\ & \cdot \langle K_{k\nu}(\mathfrak{z})K_{k\nu}(0)\rangle \\ & + \sum_{\nu=1}^{n} X_{\nu\nu} \int_{-\infty}^{+\infty} d\mathfrak{z} \exp[i(\beta_k - \beta_\nu)\mathfrak{z}] \\ & \cdot \langle K_{k\nu}(\mathfrak{z})K_{k\nu}^*(0)\rangle, \qquad (k = 1,2,\cdots,n)\end{aligned} \quad (15)$$

and

$$\begin{aligned}(d/dz)&X_{k\mu}(z) \\ &= X_{k\mu}\Bigg\{\sum_{\nu=1}^{n} \int_{0}^{\infty} d\mathfrak{z} \exp[i(\beta_\mu - \beta_\nu)\mathfrak{z}]\langle K_{\mu\nu}(\mathfrak{z})K_{\mu\nu}(0)\rangle \\ &+ \int_{-\infty}^{+\infty} d\mathfrak{z}\langle K_{\mu\mu}(\mathfrak{z})K_{kk}^*(0)\rangle \\ &+ \sum_{\nu=1}^{n} \int_{0}^{\infty} d\mathfrak{z} \exp[i(\beta_\nu - \beta_k)\mathfrak{z}]\langle K_{k\nu}^*(\mathfrak{z})K_{k\nu}^*(0)\rangle\Bigg\}, \\ &(k \neq \mu = 1,2,\cdots,n)\end{aligned} \quad (16)$$

where use has been made of (5) and of the relation [4]

$$K_{\mu\nu}(\mathfrak{z}) = K_{\nu\mu}(\mathfrak{z}). \quad (17)$$

[2] In testing this assumption, one has to be careful when considering the approximately degenerate modes introduced by Gloge [5].

Equation (15) has also been obtained by Marcuse [1], [2] and can be written in the form

$$d\langle P_k(z)\rangle/dz = \sum_{\nu=1}^{n} h_{k\nu}[\langle P_\nu(z)\rangle - \langle P_k(z)\rangle], \qquad (k = 1,2,\cdots,n) \quad (18)$$

where $\langle P_\nu \rangle \equiv X_{\nu\nu}$ labels the average power contained in the νth mode and

$$h_{k\nu} = \int_{-\infty}^{+\infty} d\mathfrak{z} \exp[i(\beta_k - \beta_\nu)\mathfrak{z}]\langle K_{k\nu}(\mathfrak{z})K_{k\nu}^*(0)\rangle = h_{\nu k} \quad (19)$$

as obtained by (15) and (17). Equations (18) and (19) entail the power conservation condition

$$(d/dz) \sum_{\nu=1}^{n} \langle P_\nu(z)\rangle = 0. \quad (20)$$

Furthermore, $h_{k\nu}$ is, in most cases, a real positive coefficient, as one can see from (19), keeping in mind that $\langle |K_{k\nu}|^2\rangle$ is a positive quantity and assuming $\langle K_{k\nu}(\mathfrak{z})K_{k\nu}^*(0)\rangle$ to be a well-behaved decreasing function of $|\mathfrak{z}|$. This assures that the asymptotic average powers

$$\langle P_k\rangle_a = \lim_{z\to\infty} \langle P_k(z)\rangle$$

fulfill the equipartition relation

$$\langle P_1\rangle_a = \langle P_2\rangle_a = \cdots = \langle P_n\rangle_a \quad (21)$$

as one obtains from (18) by imposing the condition

$$\lim_{z\to\infty} (d/dz)\langle P_k(z)\rangle = 0, \qquad \text{for all } k.$$

As an example, we write down the solution of (18) for $n = 2$, which reads

$$\begin{aligned}\langle P_1(z)\rangle &= \tfrac{1}{2}[P_1(0) + P_2(0)] + \tfrac{1}{2}[P_1(0) - P_2(0)] \exp(-2h_{12}z) \\ \langle P_2(z)\rangle &= \tfrac{1}{2}[P_1(0) + P_2(0)] - \tfrac{1}{2}[P_1(0) - P_2(0)] \exp(-2h_{12}z)\end{aligned} \quad (22)$$

and satisfies both (20) and (21).

The system of (16) for the correlation terms is particularly simple, since the $X_{k\mu}$, $(k \neq \mu)$, are not coupled among themselves. For $n = 2$, the solution reads

$$X_{12}(z) = X_{12}(0) \exp(-\alpha z - i\beta z) \quad (23)$$

where α and β are the real positive quantities

$$\alpha = h_{12} + \tfrac{1}{2}(h_{11} + h_{22}) + \int_{-\infty}^{+\infty} d\mathfrak{z}\langle K_{11}(\mathfrak{z})K_{22}(0)\rangle$$

$$\beta = 2\int_{0}^{\infty} d\mathfrak{z} \sin[(\beta_2 - \beta_1)\mathfrak{z}]\langle K_{12}^*(\mathfrak{z})K_{12}(0)\rangle. \quad (24)$$

In the general case $n > 2$, the amplitude correlation is of the type given in (23), which entails that, if no power is present at $z = 0$ in the kth and μth modes, $X_{k\mu}(z) = 0$ for all z. This implies that no information on the relative phase of c_k and c_μ is contained in the statistical theory, so that it is not possible to make any prevision on the influence of mode coupling on an interference experiment between the kth and μth modes in a single fiber.[3]

III. COUPLED EQUATIONS FOR POWER PRODUCTS

The evolution of a four-amplitude product of the type $c_\alpha^* c_\beta c_k^* c_\mu$ can be derived from (1) in the form

$$(d/dz)(c_\alpha^* c_\beta c_k^* c_\mu) = c_\alpha^* c_\beta (d/dz)(c_k^* c_\mu) + c_k^* c_\mu (d/dz)(c_\alpha^* c_\beta)$$
$$= \sum_{\nu=1}^{n} A_{\mu\nu} c_\alpha^* c_\beta c_k^* c_\nu + \sum_{\nu=1}^{n} A_{k\nu}^* c_\alpha^* c_\beta c_\nu^* c_\mu + \sum_{\nu=1}^{n} A_{\beta\nu} c_\alpha^* c_\nu c_k^* c_\mu + \sum_{\nu=1}^{n} A_{\alpha\nu}^* c_\nu^* c_\beta c_k^* c_\mu,$$
$$(\alpha, \beta, k, \mu = 1,2,\cdots,n). \tag{25}$$

We now follow a procedure analogous to that of Section II, that is, after a formal integration of (25), we average both sides of the obtained equations within a relative approximation of the order $0(nKz_0)$, thus obtaining the system of differential equations

$$(d/dz)\langle c_\alpha^* c_\beta c_k^* c_\mu\rangle = \sum_{\lambda=1}^{n}\sum_{\nu=1}^{n}\Big\{\langle c_\alpha^* c_\beta c_k^* c_\lambda\rangle \exp[i(\beta_\mu - \beta_\lambda)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\lambda - \beta_\nu)\mathfrak{z}]\langle K_{\nu\lambda}(\mathfrak{z})K_{\mu\nu}(0)\rangle$$
$$+ \langle c_\alpha^* c_\beta c_\lambda^* c_\nu\rangle \exp[i(\beta_\mu - \beta_\nu + \beta_\lambda - \beta_k)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_k - \beta_\lambda)\mathfrak{z}]\langle K_{k\lambda}^*(\mathfrak{z})K_{\mu\nu}(0)\rangle$$
$$+ \langle c_\alpha^* c_\lambda c_k^* c_\nu\rangle \exp[i(\beta_\mu - \beta_\nu + \beta_\beta - \beta_\lambda)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\lambda - \beta_\beta)\mathfrak{z}]\langle K_{\beta\lambda}(\mathfrak{z})K_{\mu\nu}(0)\rangle$$
$$+ \langle c_\lambda^* c_\beta c_k^* c_\nu\rangle \exp[i(\beta_\mu - \beta_\nu + \beta_\lambda - \beta_\alpha)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\alpha - \beta_\lambda)\mathfrak{z}]\langle K_{\alpha\lambda}^*(\mathfrak{z})K_{\mu\nu}(0)\rangle\Big\}$$
$$+ \text{same terms with the exchange } \{\beta \leftrightarrow \mu\}$$
$$+ \sum_{\lambda=1}^{n}\sum_{\nu=1}^{n}\Big\{\langle c_\alpha^* c_\beta c_\nu^* c_\lambda\rangle \exp[i(\beta_\nu - \beta_k + \beta_\mu - \beta_\lambda)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\lambda - \beta_\mu)\mathfrak{z}]\langle K_{k\nu}^*(\mathfrak{z})K_{\mu\lambda}(0)\rangle$$
$$+ \langle c_\alpha^* c_\beta c_\lambda^* c_\mu\rangle \exp[i(\beta_\lambda - \beta_k)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\nu - \beta_\lambda)\mathfrak{z}]\langle K_{k\nu}^*(\mathfrak{z})K_{\nu\lambda}^*(0)\rangle$$
$$+ \langle c_\alpha^* c_\lambda c_\nu^* c_\mu\rangle \exp[i(\beta_\nu - \beta_k + \beta_\beta - \beta_\lambda)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\lambda - \beta_\beta)\mathfrak{z}]\langle K_{k\nu}^*(\mathfrak{z})K_{\beta\lambda}(0)\rangle$$
$$+ \langle c_\lambda^* c_\beta c_\nu^* c_\mu\rangle \exp[i(\beta_\nu - \beta_k + \beta_\lambda - \beta_\alpha)z] \cdot \int_0^\infty d\mathfrak{z} \exp[i(\beta_\alpha - \beta_\lambda)\mathfrak{z}]\langle K_{k\nu}^*(\mathfrak{z})K_{\alpha\lambda}^*(0)\rangle\Big\}$$
$$+ \text{same terms with the exchange } \{k \leftrightarrow \alpha\}. \tag{26}$$

If we put $\alpha = \mu$, $\beta = k$, and remember that $|c_\mu|^2$ represents the power contained in the μth mode, we derive from (26) the closed system of coupled equations

$$(d/dz)\langle P_\mu^2(z)\rangle = 2\sum_{\nu\neq\mu}^{1,n} h_{\mu\nu}[2\langle P_\mu(z)P_\nu(z)\rangle - \langle P_\mu^2(z)\rangle], \quad (\mu = 1,2,\cdots,n) \tag{27}$$

$$(d/dz)\langle P_k(z)P_\mu(z)\rangle = -2h_{k\mu}\langle P_k(z)P_\mu(z)\rangle + \sum_{\nu=1}^{n} h_{\mu\nu}[\langle P_k(z)P_\nu(z)\rangle - \langle P_k(z)P_\mu(z)\rangle] + \sum_{\nu=1}^{n} h_{k\nu}[\langle P_\mu(z)P_\nu(z)\rangle - \langle P_k(z)P_\mu(z)\rangle], \quad (k \neq \mu = 1,2,\cdots,n) \tag{28}$$

which are valid within the "rotating wave approximation" previously used in neglecting terms not verifying either (13) or (14). The set of (27), already derived by means of the perturbative approach [2], and (28) fulfills the power conservation condition in the form

$$(d/dz)\langle[\sum_{\nu=1}^{n} P_\nu(z)]^2\rangle = 0. \tag{29}$$

It is now possible to determine for the asymptotic quantities

$$\langle P_k^2\rangle_a = \lim_{z\to\infty}\langle P_k^2(z)\rangle$$
$$\langle P_k P_\mu\rangle_a = \lim_{z\to\infty}\langle P_k(z)P_\mu(z)\rangle, \quad (k \neq \mu = 1,2,\cdots,n) \tag{30}$$

the following relation which is obtained by imposing the vanishing of the left-hand side of (27) and (28):

[3] While in most practical problems the attention is focused on the behavior of single-mode power, the relative phases of the various modes are relevant in measurements of the spatial and temporal coherence of the transmitted field [6].

$$\langle P_1^2\rangle_a = \langle P_2^2\rangle_a = \cdots = \langle P_n^2\rangle_a = 2\langle P_k P_\mu\rangle_a, \qquad (k \neq \mu = 1,2,\cdots,n). \quad (31)$$

In order to obtain information on the statistical power distribution after long distances, which in particular can furnish a criterion for the reliability of the previous results when applied to a single fiber, we note that from (21) and (20) it follows that

$$\langle P_k\rangle_a = n^{-1} \sum_{\nu=1}^{n} \langle P_\nu\rangle_a = n^{-1} \sum_{\nu=1}^{n} P_\nu(0), \qquad (k = 1,2,\cdots,n) \quad (32)$$

while (31) and (29) furnish

$$\langle P_k^2\rangle_a = [n + n(n-1)/2]^{-1}\langle [\sum_{\nu=1}^{n} P_\nu]^2\rangle_a = [n + n(n-1)/2]^{-1}[\sum_{\nu=1}^{n} P_\nu(0)]^2, \qquad (k = 1,2,\cdots,n). \quad (33)$$

By comparing (32) and (33) we obtain

$$\frac{\langle P_k^2\rangle_a}{\langle P_k\rangle_a^2} = \frac{2n}{n+1}, \qquad (k = 1,2,\cdots,n) \quad (34)$$

which also yields, with the help of (31),

$$\frac{\langle P_k P_\mu\rangle_a}{\langle P_k\rangle_a \langle P_\mu\rangle_a} = \frac{n}{n+1}, \qquad (k \neq \mu = 1, 2,\cdots, n). \quad (35)$$

If we introduce the power fluctuation of the single fiber defined as $P_k(z) = \langle P_k(z)\rangle + \Delta P_k(z)$, we can write (34) and (35) in the form

$$\langle \Delta P_k^2\rangle_a = \frac{n-1}{n+1} \langle P_k\rangle_a^2, \qquad (k = 1,2,\cdots,n) \quad (36)$$

$$\langle \Delta P_k \Delta P_\mu\rangle_a = -\langle P_k\rangle_a \langle P_\mu\rangle_a (n+1)^{-1}, \qquad (k \neq \mu = 1,2,\cdots,n). \quad (37)$$

For a large number of modes, we obtain from (36) the following relation:

$$\frac{\langle \Delta P_k^2\rangle_a}{\langle P_k^2\rangle_a} = 1, \qquad (k = 1,2,\cdots,n) \quad (38)$$

which is consistent with an exponential intensity distribution and agrees with the results of [2], in which the cross-correlation terms given in (37) are neglected *a priori*.

The asymptotic "normalized variance" decreases for a small number of coupled modes and reduces to

$$\frac{\langle \Delta P_k^2\rangle_a}{\langle P_k^2\rangle_a} = \frac{1}{3}, \qquad (k = 1,2,\cdots,n) \quad (39)$$

for $n = 2$. We finally observe that the decreasing importance of the cross-correlation terms, when n becomes large, is clearly shown by (37).

IV. CONCLUSIVE REMARKS

We have used a simple statistical method in order to evaluate the coupling between different modes propagating in an optical fiber. We have considered an ensemble of similar fibers in which the effects of mode coupling predominate over those of power losses. That is a case of increasing practical interest [7], [8].

The main contributions of this paper consist of the evaluation of the cross correlation between powers of different modes and in the determination of the asymptotic normalized variance of single-mode power, in the general case of a finite number of coupled modes.

REFERENCES

[1] a) D. Marcuse, "Derivation of coupled power equations," *Bell Syst. Tech. J.*, vol. 51, pp. 229–237, Jan. 1972. (For a better derivation, see the following.)
b) ——, *Theory of Dielectric Optical Waveguides*. New York: Academic, 1974, ch. 5.

[2] ——, "Fluctuation of the power of coupled modes," *Bell Syst. Tech. J.*, vol. 51, pp. 1793–1800, Oct. 1972.

[3] a) A. T. Bharucha-Reid, Ed. *Probabilistic Methods in Applied Mathematics*, vols. I and II. New York: Academic, 1968 and 1970.
b) J. McKenna and J. A. Morrison, "Moments and correlation functions of solutions of a stochastic differential equation," *J. Math Phys.*, vol. 11, pp. 2348–2360, Aug. 1970.
c) ——, "Moments of solutions of a class of stochastic differential equations," *J. Math. Phys.*, vol. 12, pp. 2126–2136, Oct. 1971.

[4] D. Marcuse, "Coupled mode theory of round optical fibers," *Bell Syst. Tech. J.*, vol. 52, pp. 817–842, July/Aug. 1973.

[5] D. Gloge, "Optical mode flow in multimode fibers," *Bell Syst. Tech. J.*, vol. 51, pp. 1767–1783, Oct. 1972.

[6] B. Crosignani and P. Di Porto, "Coherence of an electromagnetic field propagating in a weakly guiding fiber," *J. Appl. Phys.*, vol. 44, pp. 4616–4617, Oct. 1973.

[7] S. D. Personick, "Time dispersion in dielectric waveguides," *Bell Syst. Tech. J.*, vol. 50, pp. 843–859, Mar. 1971.

[8] S. E. Miller, E. A. J. Marcatili, and T. Li, "Research toward optical-fiber transmission systems," *Proc. IEEE*, vol. 61, pp. 1703–1751, Dec. 1973.